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# Renormalization group treatment of nonrenormalizable interactions

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## Abstract

The structure of the UV divergences in higher dimensional nonrenormalizable theories is analysed. Based on renormalization operation and renormalization group theory it is shown that even in this case the leading divergences (asymptotics) are governed by the one-loop diagrams the number of which, however, is infinite. An explicit expression for the one-loop counter term in an arbitrary  $D$ -dimensional quantum field theory without derivatives is suggested. This allows one to sum up the leading asymptotics which are independent of the arbitrariness in subtraction of higher order operators. Diagrammatic calculations in a number of scalar models in higher loops are performed to be in agreement with the above statements. These results do not support the idea of the naïve power-law running of couplings in nonrenormalizable theories and fail (with one exception) to reveal any simple closed formula for the leading terms.

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## 1. Introduction

It has been commonly accepted that one cannot use nonrenormalizable interactions beyond the tree level due to uncontrollable ultraviolet (UV) divergences. Nothing has changed in understanding of this problem; however, these days it is sometimes suggested that nonrenormalizable interactions are treated on equal footing with the renormalizable ones. This specially became fashionable within the context of extra-dimensional theories, all of them being nonrenormalizable in a usual sense. A wide spread opinion ensures these theories to be treated as effective ones [1], meaning that one believes that the UV troubles are cured somehow by including these theories in a more general framework, for instance, a string theory

while considering the low energy effective action. In the latter case, one distinguishes the ‘relevant’, ‘marginal’ and ‘irrelevant’ operators in the manner of Wilson [2], so that at low energies one can abandon contributions from irrelevant operators being power suppressed and end up with relevant and marginal operators which are renormalizable.

However, this is not the case of higher dimensional theories since there are no relevant or marginal operators there. They are all irrelevant or nonrenormalizable and one cannot throw away all of them. Hence, one has to find the way to deal with them or to give up.

Evidently there are no problems at the tree level, where one can drop the contributions from the higher order operators, but in the loops one is faced with severe properties of nonrenormalizable interactions: appearance of an infinite sequence of higher order operators. This is true even at low energies (smaller than an intrinsic scale set up by dimensional couplings).

Sometimes one talks about ‘low energy renormalizability’ of nonrenormalizable theories assuming the ignorance of the contributions of these higher order operators. Within this context one discusses the ‘power-law running’ of couplings [3]. This seemingly an attractive idea needs a thorough investigation.

In higher dimensional theories (meaning the dimension is higher than the critical dimension of a given interaction which is usually 4) one can approach this problem in two ways: either to consider the theory directly in (flat) extra dimensions and try to cope with power-like divergences, or to use the kind of Kaluza–Klein approach with (compact) extra dimensions and consider four-dimensional theory with an infinite tower of K–K modes [4]. In the first case, one faces the problem of appearance of new higher dimensional operators as the UV counter terms, and in the latter one, the problem of divergence of the sums over the K–K states. Nevertheless, it is claimed that at low energies when we ignore (in a sense of effective theory) the higher order operators or cuts the divergence of the sum introducing a cut-off momentum, we get the power-like behaviour of the Green functions or the power-law running of the original couplings [3]. Similar results follow from the renormalization group (RG) approach in the manner of Wilson based on the  $\epsilon$ -expansion and analytical continuation of the perturbation theory expansion above the critical dimension. In this case, one has a nonperturbative fixed point where the theory possesses conformal invariance so that the effective coupling becomes dimensionless that is sometimes referred to as nonperturbative renormalizability [5, 6]. Here, the higher order operators are suppressed in the infrared domain and the Green functions in the vicinity of the fixed point exhibit the power-like behaviour [7].

However, the running literally means the summation of the leading asymptotics (the leading logs or leading powers). Hence, assuming these considerations to be correct, one sums up the leading contributions of an infinite sequence of diagrams into the ‘running’ quantities. If this is true, at least at low energies, one has to be able to check by explicit calculations of diagrams how the leading terms are summed up. This means that they are essentially predicted prior to the calculation. It is very well known how it works in renormalizable theories within the renormalization group approach. The question is whether it also works in the nonrenormalizable case.

The purpose of this paper is to demonstrate that indeed the structure of local quantum field theory (QFT) even in nonrenormalizable theories reveals the one-loop origin of the leading asymptotics despite the fact that there is no simple closed expression for the amplitudes like in the renormalizable case. Our results, including explicit calculation of the diagrams in some nonrenormalizable models, do not support the idea of the naïve ‘power-law running’ of couplings and lead (with one miraculous exception) to complicated expressions without obvious summation pattern.

## 2. Renormalization operation in local QFT

The essence of the diagram behaviour can be figured out from the structure of the so-called R(enormalization)-operation valid in any local QFT. It basically states that genuine UV divergences even in nonrenormalizable theories are always local (or quasi-local), i.e. contain a limited number of derivatives. So the bare Lagrangian being properly regularized contains all possible local counter terms. In their turn, these counter terms are in one-to-one correspondence with the logarithms that appear on top of the powers of momenta.

Since the bare Lagrangian does not depend on the renormalization scale, the explicit dependence of the counter terms on the scale has to be compensated by the inexplicit dependence of the couplings. In the case of a renormalizable Lagrangian, there is one or a few couplings, and differentiating the bare Lagrangian with respect to a scale one gets the corresponding RG equations. For the dimensional regularization one gets the so-called pole equations [8] that relate the higher order poles with the lowest one in all orders of perturbation theory. In particular, taking the one-loop contribution to the lowest pole one recovers the whole infinite series of the leading pole terms. In the case of a single coupling they are summed into a geometric progression

$$g^{\text{bare}} = (\mu^2)^\varepsilon \frac{g}{1 + bg/\varepsilon}, \quad (1)$$

where the coefficient  $b$  comes from the one-loop  $\beta$ -function. For several couplings the expressions are more complicated but the higher order terms can still be summed up and are completely defined by the one-loop contribution.

For the nonrenormalizable Lagrangian, the R-operation still holds, but now one has an infinite number of terms in the bare Lagrangian. Even if one starts with the finite number of terms new higher order operators will be generated via the UV counter terms. Moreover, since the couplings in the nonrenormalizable case are dimensionful (as it always happens in extra dimensions), the diagrams reveal only higher dimensional operators to compensate the negative dimension of the coupling. The original operators do not appear. So the counter terms possess the triangle structure: each operator gives a contribution to the renormalization of the higher operators and not to itself and lower ones. Still, despite this cumbersome picture, it has the same structure as in renormalizable theories. The higher order poles are still defined by the lowest one and the leading ones are determined by the one-loop diagrams. The corresponding pole equations for a general QFT theory were written in [9]. We present them here without a derivation:

$$\mathcal{L}^{\text{Bare}} = (\mu^2)^\varepsilon \left( \mathcal{L} + \sum_{n=1}^{\infty} \frac{A_n(\mathcal{L})}{\varepsilon^n} \right), \quad (2)$$

$$\left( \mathcal{L} \frac{\delta}{\delta \mathcal{L}} - 1 \right) A_n(\mathcal{L}) = \beta(\mathcal{L}) \frac{\delta}{\delta \mathcal{L}} A_{n-1}(\mathcal{L}), \quad \beta(\mathcal{L}) = \left( \mathcal{L} \frac{\delta}{\delta \mathcal{L}} - 1 \right) A_1(\mathcal{L}), \quad (3)$$

where  $A_n(\mathcal{L})$  means that the corresponding counter term is calculated starting from the Lagrangian  $\mathcal{L}$ .

Performing loop expansion and taking into account that the counter terms  $A_n(\mathcal{L})$  are homogeneous functions

$$A_n(\mathcal{L}) = \sum_{k=n}^{\infty} A_{nk}(\mathcal{L}), \quad A_{nk}(\lambda \mathcal{L}) = \lambda^k A_{nk}(\mathcal{L})$$

(here the first subscript denotes the order of the pole term, and the second one, the number of loops), one can reduce equation (3) to

$$\left(\lambda \frac{\delta}{\delta\lambda} - 1\right) A_n(\lambda\mathcal{L})|_{\lambda=1} = \frac{d}{d\eta} A_{n-1}(\mathcal{L} + \eta\beta(\mathcal{L}))|_{\eta=0} \quad (4)$$

$$\beta(\mathcal{L}) = \left(\lambda \frac{\delta}{\delta\lambda} - 1\right) A_1(\lambda\mathcal{L})|_{\lambda=1} = \sum_{k=n}^{\infty} k A_{1k}(\mathcal{L}). \quad (5)$$

For the leading poles this leads to

$$A_{nn}(\mathcal{L}) = \frac{1}{n} \frac{d}{d\eta} A_{n-1n-1}(\mathcal{L} + \eta A_{11}(\mathcal{L}))|_{\eta=0}. \quad (6)$$

One can clearly see from equation (6) that if one knows the one-loop contribution to the simple pole, namely  $A_{11}$ , then one knows via a recursive procedure all the leading terms  $A_{nn}$ . We would like to stress once again that this statement and equation (6) are true in any theory, renormalizable or nonrenormalizable.

However, there is some crucial difference in application of this recursion to the renormalizable or nonrenormalizable theories. While in the renormalizable theories one has a finite number of one-loop diagrams contributing to  $A_{11}$  that are constructed from the original Lagrangian, in the nonrenormalizable case in general one has an infinite number of such one-loop diagrams involving new higher dimensional operators. So unless one has some hint how these diagrams are related to one another, one has an infinite number of unknown coefficients in  $A_{11}$ .

Consider the scalar field theories when the original interaction Lagrangian does not contain derivatives. In four dimensions the explicit closed expression for  $A_{11}$  has the form [10]

$$A_{11}(\mathcal{L}) = -\frac{1}{(4\pi)^2} \frac{1}{4} \frac{\delta^2 \mathcal{L}}{\delta\phi^2} \times \frac{\delta^2 \mathcal{L}}{\delta\phi^2}. \quad (7)$$

Natural generalization of this formula to  $D$  dimensions would be

$$A_{11}(\mathcal{L}) = -\frac{1}{(4\pi)^{D/2}} \frac{\Gamma(D/2 - 1)}{4\Gamma(D - 2)} \frac{\delta^2 \mathcal{L}}{\delta\phi^2} (\partial^2)^{D/2-2} \frac{\delta^2 \mathcal{L}}{\delta\phi^2}. \quad (8)$$

However, when  $D > 4$  there are other one-loop divergent diagrams having more external legs. By taking into account higher dimensional operators which appear in higher loops the number of these terms increases. To reproduce this increasing sequence of terms we conjecture that in  $D$  dimensions the one-loop counter term has the form

$$\begin{aligned} A_{11}(\mathcal{L}) &= -\frac{1}{(4\pi)^{D/2}} \frac{\Gamma(D/2 - 1)}{4\Gamma(D - 2)} \frac{\delta^2 \mathcal{L}}{\delta\phi^2} \frac{(\partial^2)^{D/2}}{\left(\partial^2 + \frac{\delta^2 \mathcal{L}}{\delta\phi^2}\right)^2} \frac{\delta^2 \mathcal{L}}{\delta\phi^2} \\ &= -\frac{1}{(4\pi)^{D/2}} \frac{\Gamma(D/2 - 1)}{4\Gamma(D - 2)} \frac{\delta^2 \mathcal{L}}{\delta\phi^2} \frac{(\partial^2)^{D/2-2}}{\left(1 + \partial^{-2} \frac{\delta^2 \mathcal{L}}{\delta\phi^2}\right)^2} \frac{\delta^2 \mathcal{L}}{\delta\phi^2}, \end{aligned} \quad (9)$$

where the denominator is understood as a geometric progression with derivatives acting on expansion terms so as to cancel all nonlocalities  $\partial^{-2}$ . Hence, for a given  $D$  one has only a finite number of terms to contribute. However, while calculating  $A_{nn}$ , according to equation (6), one has to replace  $\mathcal{L}$  by  $A_{11}$ , which may contain extra derivatives. These extra derivatives can also cancel the  $\partial^{-2}$  terms, so the expansion goes further. The *general recipe* is: use

equation (6) for  $A_{nm}$  with  $A_{11}(\mathcal{L})$  given by equation (9) and expand the denominator until the nonlocal terms are cancelled by the corresponding derivatives. Surely, at a given order of perturbation theory only a finite number of expansion terms contribute.

Below we illustrate these statements considering calculation of the leading order terms explicitly in a number of models. These calculations prove that equation (6) is valid, but the numbers obtained (with one exception) do not reveal any obvious summation pattern.

### 3. Explicit calculations in nonrenormalizable models

For the sake of simplicity we consider a set of massless scalar field theories within the framework of dimensional regularization. When the coupling  $g$  in a given dimension  $D$  has a negative dimension, the theory is nonrenormalizable in the usual sense.

#### 3.1. $D = 6, \phi_{(6)}^4$

Let us start with the Lagrangian

$$\mathcal{L}_{\text{int}} = -\frac{\lambda\phi^4}{4!} \quad (10)$$

in  $D = 6 - 2\varepsilon$ . The coupling  $\lambda$  has a negative dimension  $[\lambda] = -2 + 2\varepsilon$  that leads to an infinite series of the counter terms containing higher order operators generated via loop expansion. One has, according to equation (9),

$$A_{11}(\mathcal{L}) = -\frac{1}{(4\pi)^3} \frac{1}{24} \mathcal{L}'' \frac{\partial^2}{(1 + \partial^{-2}\mathcal{L}'')^2} \mathcal{L}'' = -\frac{1}{(4\pi)^3} \frac{1}{24} \left( \frac{\lambda^2}{4} \phi^2 \partial^2 \phi^2 + \frac{\lambda^3}{4} (\phi^2)^3 \right), \quad (11)$$

where following the above-mentioned recipe we omitted the nonlocal terms. Hereafter we use the notation  $\mathcal{L}'' \equiv \frac{\delta^2 \mathcal{L}}{\delta \phi^2}$ . Two terms of the expansion in equation (11) correspond to the one-loop two-point and triangle diagrams, respectively.

If one substitutes equation (9) into (6), one gets for  $A_{22}$

$$2A_{22}(\mathcal{L}) = -\frac{1}{(4\pi)^3} \frac{1}{24} \left\{ [A_{11}(\mathcal{L})]'' \frac{\partial^2}{(1 + \partial^{-2}\mathcal{L}'')^2} \mathcal{L}'' - 2\mathcal{L}'' \frac{[A_{11}(\mathcal{L})]''}{(1 + \partial^{-2}\mathcal{L}'')^3} \mathcal{L}'' + \mathcal{L}'' \frac{\partial^2}{(1 + \partial^{-2}\mathcal{L}'')^2} [A_{11}(\mathcal{L})]'' \right\}, \quad (12)$$

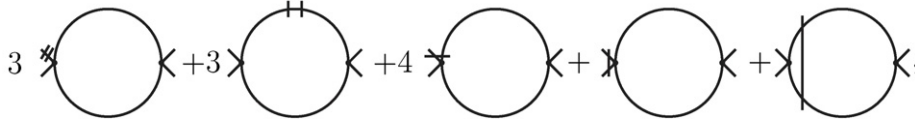
where  $A_{11}$  is given by equation (11). Substituting (11) into (12) and performing the expansion of the geometric progression, one finally gets

$$A_{22} = -\left( \frac{1}{(4\pi)^3} \frac{1}{24} \right)^2 \left\{ \frac{\lambda^3}{8} (\phi^2 \partial^2 \phi^2)'' \partial^2 \phi^2 + \frac{\lambda^4}{8} [(\phi^2)^3]'' \partial^2 \phi^2 + \frac{3\lambda^4}{16} (\phi^2 \partial^2 \phi^2)'' (\phi^2)^2 + \frac{3\lambda^5}{16} [(\phi^2)^3]'' (\phi^2)^2 + \frac{3\lambda^5}{16} (\phi^2)^2 (\phi^2 \partial^2 \phi^2)'' \partial^{-2} \phi^2 \right\}. \quad (13)$$

The tricky point here is the interference of the variational derivative with respect to  $\phi$  and the ordinary spacetime derivative. This requires some definition. Let us first evaluate the spacetime derivatives and then the variational derivatives

$$\begin{aligned} (\phi^2 \partial^2 \phi^2)'' &= 2(\phi^3 \partial^2 \phi + \phi^2 (\partial \phi)^2)'' \\ &= 4(3\phi \partial^2 \phi + 3\phi^2 \partial^2 + 4\phi \partial \phi \partial + (\partial \phi)^2 + \phi^2 \partial \partial). \end{aligned} \quad (14)$$

When the spacetime derivative is left free, we understand it as acting on the propagator of the corresponding one-loop diagram. Schematically it is shown below



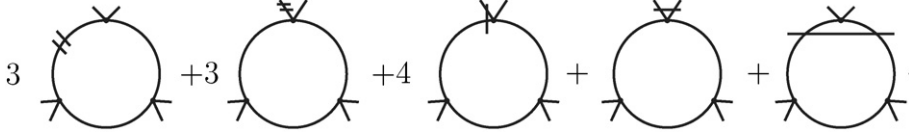
where the crossed lines mean the corresponding derivatives.

The next step is the reduction of the diagram with derivative(s) to the one without them. Usually this step is straightforward and is performed by analysing the corresponding one-loop integral. In this particular case, one has

$$\begin{aligned}
 (\phi^2 \partial^2 \phi^2)'' \times \text{diagram} &\Rightarrow 4 \left( 3p_1^2 + 3 * 0 - \frac{4pp_1}{2} + p_1 p_2 + \frac{p^2}{2} \right) \phi\phi \\
 &= 8p_1^2 \phi\phi = 8\phi \partial^2 \phi,
 \end{aligned} \tag{15}$$

where  $p$  is the momentum entering the loop, and  $p_1$  and  $p_2$  are the momenta of each leg. Here we take the symmetrical point where  $p^2 = \frac{4}{3}p_1^2 = \frac{4}{3}p_2^2$ ,  $p_1 p_2 = -\frac{1}{3}p_1^2$ .

Now let us take the third term in (13). Here, we have triangle diagrams

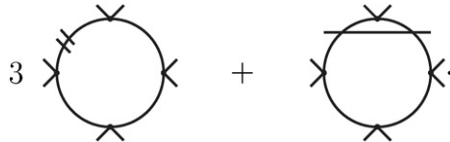


Again reducing the diagram with derivatives to the one without them one obtains

$$(\phi^2 \partial^2 \phi^2)'' \times \text{diagram} \Rightarrow 4 \left( 3 \left( -\frac{p_{12}^2}{3} \right) + 3p_1^2 - \frac{8}{5}p_1^2 + \frac{5}{6}p_{12}^2 + \frac{pp_1}{2} \right) \phi\phi = \frac{7\lambda^4}{10} \phi \partial^2 \phi,$$

where  $p_1$  and  $p_2$  are the momenta in each leg and  $p_{12}$  is the momentum entering in one vertex. Here we take the symmetrical point where  $p_1 p_2 = -\frac{1}{5}p_1^2$  and  $p_{12}^2 = \frac{8}{5}p_1^2$ .

We are left with the last term in equation (13). Here one has the box diagram which is convergent in  $D = 6$  if there are no derivatives. This means that only the second and the last terms of equation (14) contribute. One has



The first diagram is a triangle one and the second is easily reduced to it. As a result one has

$$(\phi^2 \partial^2 \phi^2)'' \times \text{diagram} \Rightarrow 4 \left( \frac{3}{2} - \frac{1}{2} \right) (-) \phi\phi = -4\phi\phi, \tag{16}$$

where the minus sign of the last term is due to the reduction of the box diagram to a triangle one.

Adding up all terms equation (13) finally leads to

$$\begin{aligned}
 A_{22} = - \left( \frac{1}{(4\pi)^3} \frac{1}{24} \right)^2 &\left\{ \lambda^3 (\phi \partial^2 \phi) \partial^2 \phi^2 + \frac{15\lambda^4}{4} (\phi^2)^2 \partial^2 \phi^2 + \frac{7\lambda^4}{10} \phi \partial^2 \phi (\phi^2)^2 \right. \\
 &\left. + \frac{45\lambda^5}{8} (\phi^2)^2 (\phi^2)^2 - \frac{3\lambda^5}{4} (\phi^2)^2 (\phi^2)^2 \right\}.
 \end{aligned} \tag{17}$$

One may proceed further and get

$$\begin{aligned}
 A_{33} &= - \left( \frac{1}{(4\pi)^3} \frac{1}{24} \right)^3 \left\{ \frac{\lambda^4}{6} [2(\phi \partial^2 \phi) \partial^2 (\phi \partial^2 \phi) + ((\phi \partial^2 \phi) \partial^2 \phi^2)'' \partial^2 \phi^2 \right. \\
 &\quad \left. + (\phi^2 \partial^2 (\phi \partial^2 \phi))'' \partial^2 \phi^2] + \dots \right\} \\
 &= - \left( \frac{1}{(4\pi)^3} \frac{1}{24} \right)^3 \left\{ \frac{2\lambda^4}{3} [2(\phi \partial^2 \phi) \partial^2 (\phi \partial^2 \phi) + \partial^2 \phi \partial^2 \phi \partial^2 \phi^2 + \phi \partial^2 \partial^2 \phi \partial^2 \phi^2] + \dots \right\}, \quad (18)
 \end{aligned}$$

where we keep only the terms quartic in fields.

To compare these expressions with explicit diagram calculation, it is useful to transfer to the momentum representation. Then the spacetime derivative means some momenta with the proper symmetrization which depends on the number of legs. Thus, equations (11), (17) and (18) become

$$\begin{aligned}
 A_{11} &= - \frac{1}{(4\pi)^3} \frac{1}{24} \left( - \frac{\lambda^2}{12} (\phi^2)^2 (s+t+u) + \frac{\lambda^3}{4} (\phi^2)^3 \right), \\
 A_{22} &= - \left( \frac{1}{(4\pi)^3} \frac{1}{24} \right)^2 \left\{ \frac{\lambda^3}{12} (s+t+u)^2 (\phi^2)^2 - \lambda^4 \frac{67}{60} (\phi^2)^3 \sum_{i=1}^6 p_i^2 + \lambda^5 \frac{39}{8} (\phi^2)^4 \right\}, \quad (19) \\
 A_{33} &= - \left( \frac{1}{(4\pi)^3} \frac{1}{24} \right)^3 \left\{ - \frac{\lambda^4}{72} [2(\phi^2)^2 + (\phi^2)^2 + (\phi^2)^2] (s+t+u)^3 + \dots \right\}.
 \end{aligned}$$

We have checked by explicit diagram calculations that equations (19) are indeed correct.

It is interesting to consider the four-point function. The Lagrangian, together with the corresponding counter terms, looks like

$$\begin{aligned}
 \phi^4 &: - \frac{\lambda}{24} + \frac{\lambda^2}{(4\pi)^3} \frac{1}{24\epsilon} \frac{s+t+u}{12} - \frac{\lambda^3}{(4\pi)^6} \frac{1}{(24\epsilon)^2} \frac{(s+t+u)^2}{12} + \frac{\lambda^4}{(4\pi)^9} \frac{1}{(24\epsilon)^3} \frac{(s+t+u)^3}{18} + \dots \\
 &= - \frac{\lambda}{24} \left[ 1 - \frac{\lambda}{(4\pi)^3 \epsilon} \frac{s+t+u}{12} + \left( \frac{\lambda}{(4\pi)^3 \epsilon} \right)^2 \left( \frac{s+t+u}{12} \right)^2 \frac{1}{2} \right. \\
 &\quad \left. - \left( \frac{\lambda}{(4\pi)^3 \epsilon} \right)^3 \left( \frac{s+t+u}{12} \right)^3 \frac{1}{6} + \dots \right] \\
 &= - \frac{\lambda}{24} \exp \left( - \frac{\lambda}{(4\pi)^3 \epsilon} \frac{s+t+u}{12} \right). \quad (20)
 \end{aligned}$$

One can see that the first three terms remarkably remind the expansion of the exponent. We have checked this fact by *explicit* calculation of the diagrams up to *four* loops and with the help of the *pole equations* (3) up to  $A_{55}$ . In the latter case, unlike the diagram calculation, it is straightforward and can be easily continued further. So far we failed to find all order proof that everything is summed up to the exponent, though it seems quite reasonable.

The pole relation (20) evidently leads to the corresponding expression for the four-point function

$$\Gamma_4 = \exp \left[ \frac{\lambda}{(4\pi)^3} \frac{s}{4} \log(s/\mu^2) \right], \quad (21)$$

where we substituted symmetric asymptotics  $s = t = u$ .



Two comments are in order.

- (1) In nonrenormalizable theories, due to the presence of an infinite series of operators one has an infinite number of normalization conditions. That is why the theory is not defined. However, the leading poles (or the leading logarithms) are independent (!) of these conditions. Thus, the leading behaviour is defined unambiguously.
- (2) Equation (21) has quite an unusual form different from the geometric progression expected from the naive ‘power-law running’. Whether it is occasional or a common feature of nonrenormalizable theories needs to be clarified.

To check this exponential behaviour we made similar calculations in several other models.

### 3.2. $\phi_{(8)}^4$ , $\phi_{(10)}^4$ and $\phi_{(D)}^4$

Let us first consider the  $D = 8$  case. According to equation (9),  $A_{11}$  takes the form

$$\begin{aligned} A_{11}(\mathcal{L}) &= -\frac{1}{(4\pi)^4} \frac{1}{240} \mathcal{L}'' \frac{\partial^4}{(1 + \partial^{-2} \mathcal{L}'')^2} \mathcal{L}'' \\ &= -\frac{1}{(4\pi)^4} \frac{1}{240} \left( \frac{\lambda^2}{4} \phi^2 \partial^4 \phi^2 + \frac{\lambda^3}{4} \partial^2 \phi^2 (\phi^2)^2 + \frac{3\lambda^4}{16} (\phi^2)^4 \right), \end{aligned} \quad (22)$$

where we again omitted the nonlocal terms. Obviously, with the increase of dimension the length of the  $A_{11}$  term increases;  $A_{22}$  then becomes

$$A_{22} = -\left( \frac{1}{(4\pi)^4} \frac{1}{240} \right)^2 \left\{ \frac{\lambda^2}{8} (\phi^2 \partial^4 \phi^2)'' \partial^4 \phi^2 + \dots \right\}, \quad (23)$$

where we kept only the terms quartic in fields.

Here again we are faced with the problem of evaluating the variational derivatives. We first evaluate the spacetime derivatives

$$\begin{aligned} (\phi^2 \partial^4 \phi^2)'' &= 4\partial^2 \phi \partial^2 \phi + 4\phi \partial^2 \partial^2 \phi + 16\partial \phi \partial \partial^2 \phi + 8\partial \partial \phi \partial \partial \phi \\ &\quad + 8\phi (2\partial^2 \phi \partial^2 + \partial^2 \partial^2 \phi + \phi \partial^2 \partial^2 + 4\partial \partial^2 \phi \partial + 4\partial \phi \partial \partial^2 + 4\partial \partial \phi \partial \partial) \\ &\quad + 2\phi^2 (4\partial \partial \partial \partial). \end{aligned} \quad (24)$$

The meaning of partial derivatives acting on the right is understood as above in a sense of acting on the lines of the one-loop diagrams. This gives

$$\begin{aligned} A_{22} &= -\left( \frac{1}{(4\pi)^4} \frac{1}{240} \right)^2 \\ &\quad \times \left\{ \frac{\lambda^2}{8} \left( \frac{20}{7} \partial^2 \phi \partial^2 \phi + 4\phi \partial^2 \partial^2 \phi + 2\partial^2 \partial^2 \phi^2 + \frac{120}{7} \partial \partial \phi \partial \partial \phi + 16\partial \phi \partial \partial^2 \phi \right) \partial^4 \phi^2 + \dots \right\}, \end{aligned} \quad (25)$$

Unlike the  $D = 6$  case, the expression for  $A_{22}$  in  $D = 8$  does not look simple. Indeed, in the momentum representation it is not expressed in terms of the Mandelstam variables  $s$ ,  $t$  and  $u$  and at the symmetrical point it looks like

$$\begin{aligned} \phi^4 &: -\frac{\lambda}{24} - \frac{\lambda^2}{(4\pi)^4} \frac{1}{240\varepsilon} \frac{s^2}{4} - \frac{\lambda^3}{(4\pi)^8} \frac{1}{(240\varepsilon)^2} \frac{55s^4}{112} + \dots \\ &= -\frac{\lambda}{24} \left( 1 + \frac{\lambda}{(4\pi)^4} \frac{s^2}{40\varepsilon} + \frac{\lambda^2}{(4\pi)^8} \frac{s^4}{(40\varepsilon)^2} \frac{55}{168} + \dots \right). \end{aligned} \quad (26)$$

This result is also confirmed by explicit diagrammatic calculation. One cannot see any trace of exponent here. Moreover, it does not look like any other simple function since, as we have already mentioned, it cannot be expressed in terms of the Mandelstam variables.

We have carried out the same calculation in  $D = 10$ . Equation (26) in this case becomes

$$\begin{aligned} \phi^4 &: -\frac{\lambda}{24} + \frac{\lambda^2}{(4\pi)^5} \frac{1}{3360\varepsilon} \frac{s^3}{4} - \frac{\lambda^3}{(4\pi)^{10}} \frac{1}{(3360\varepsilon)^2} \frac{91s^6}{192} + \dots \\ &= -\frac{\lambda}{24} \left( 1 - \frac{\lambda}{(4\pi)^5} \frac{s^3}{560\varepsilon} + \frac{\lambda^2}{(4\pi)^{10}} \frac{s^6}{(560\varepsilon)^2} \frac{91}{288} + \dots \right). \end{aligned} \quad (27)$$

Again one can see no trace of exponent. To understand it better, we have calculated the second-order term in the case of  $\phi^4$  theory in  $D$  dimensions. The simplest way it can be done is by explicit diagram calculation. The result is

$$\begin{aligned} \phi^4 &: -\frac{\lambda}{24} \left\{ 1 + \frac{\lambda}{(4\pi)^{D/2}} \frac{(-1)^{D/2} 3\Gamma(D/2 - 1)}{2\Gamma(D - 2)} \frac{s^{D/2-2}}{\varepsilon} \right. \\ &\quad \left. + \left( \frac{\lambda}{(4\pi)^{D/2}} \frac{(-1)^{D/2} 3\Gamma(D/2 - 1)}{2\Gamma(D - 2)} \frac{s^{D/2-2}}{\varepsilon} \right)^2 c(D) + \dots \right\}, \end{aligned} \quad (28)$$

where

$$c(D) = \frac{1}{3} + \frac{2^{5-D}}{3^{3-D/2}} F_{21} \left( \frac{3D}{2} - 4, 2 - \frac{D}{2}; \frac{D-1}{2} \middle| \frac{1}{3} \right).$$

This complicated expression is remarkably simplified for particular values of  $D$  and at low dimensions it is

$c$	$D = 4$	$D = 6$	$D = 8$	$D = 10$	$D = 12$	$D \rightarrow \infty$
	1	1/2	55/168	91/288	6005/18 304	$\rightarrow 1/3$

#### 4. Conclusion

Summarizing we would like to stress once again that

- (1) direct calculations of Feynman diagrams demonstrate all characteristic features and problems of nonrenormalizable interactions;
- (2) for nonrenormalizable interactions like for renormalizable ones the leading divergences (asymptotics) are defined by the one-loop diagrams;
- (3) in the nonrenormalizable case contrary to the renormalizable one the number of one-loop diagrams is infinite;
- (4) we conjectured the form of the one-loop counter term in arbitrary scalar QFT with derivativeless interactions and checked it by explicit calculations in lower loops;
- (5) using the renormalization group technique, it is possible to sum up the leading asymptotics which are independent of the arbitrariness in subtraction of higher order operators;
- (6) unlike the renormalizable case, this summation does not reveal (with one exception so far) any simple function; and
- (7) the naïve power-law running seems not to be valid.

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